

LOCAL MAXIMAL OPERATORS ON FRACTIONAL SOBOLEV SPACES

HANNES LUIRO AND ANTTI V. VÄHÄKANGAS

ABSTRACT. In this note we establish the boundedness properties of local maximal operators M_G on the fractional Sobolev spaces $W^{s,p}(G)$ whenever G is an open set in \mathbb{R}^n , $0 < s < 1$ and $1 < p < \infty$. As an application, we characterize the fractional (s, p) -Hardy inequality on a bounded open set G by a Maz'ya-type testing condition localized to Whitney cubes.

1. INTRODUCTION

The local Hardy–Littlewood maximal operator $M_G = f \mapsto M_G f$ is defined for an open set $\emptyset \neq G \subsetneq \mathbb{R}^n$ and a function $f \in L^1_{\text{loc}}(G)$ by

$$M_G f(x) = \sup_r \int_{B(x,r)} |f(y)| dy, \quad x \in G,$$

where the supremum ranges over $0 < r < \text{dist}(x, \partial G)$. Whereas the (local) Hardy–Littlewood maximal operator is often used to estimate the absolute size, its Sobolev mapping properties are perhaps less known. The classical Sobolev regularity of M_G is established by Kinnunen and Lindqvist in [11]; we also refer to [5, 9, 12, 13, 15]. Concerning smoothness of fractional order, the first author established in [16] the boundedness and continuity properties of M_G on the Triebel–Lizorkin spaces $F^s_{pq}(G)$ whenever G is an open set in \mathbb{R}^n , $0 < s < 1$ and $1 < p, q < \infty$.

Our main focus lies in the mapping properties of M_G on a fractional Sobolev space $W^{s,p}(G)$ with $0 < s < 1$ and $1 < p < \infty$, cf. Section 2 for the definition or [1] for a survey of this space. The intrinsically defined function space $W^{s,p}(G)$ on a given domain G coincides with the trace space $F^s_{pp}(G)$ if and only if G is regular, i.e.,

$$|B(x, r) \cap G| \simeq r^n$$

whenever $x \in G$ and $0 < r < 1$, see [21, Theorem 1.1] and [20, pp. 6–7]. As a consequence, if G is a regular domain then M_G is bounded on $W^{s,p}(G)$. Moreover, the following question arises: is M_G a bounded operator on $W^{s,p}(G)$ even if G is not regular, e.g., if G has an exterior cusp? Our main result provides an affirmative answer to the last question:

Theorem 1.1. *Let $\emptyset \neq G \subsetneq \mathbb{R}^n$ be an open set, $0 < s < 1$ and $1 < p < \infty$. Then, there is a constant $C = C(n, p, s) > 0$ such that inequality*

$$\int_G \int_G \frac{|M_G f(x) - M_G f(y)|^p}{|x - y|^{n+sp}} dy dx \leq C \int_G \int_G \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dy dx \quad (1.1)$$

holds for every $f \in L^p(G)$. In particular, the local Hardy–Littlewood maximal operator M_G is bounded on the fractional Sobolev space $W^{s,p}(G)$.

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The relatively simple proof of Theorem 1.1 is based on a pointwise inequality in \mathbb{R}^{2n} , see Proposition 3.1. That is, for $f \in L^p(G)$ we define an auxiliary function $S(f) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$

$$S(f)(x, y) = \frac{\chi_G(x)\chi_G(y)|f(x) - f(y)|}{|x - y|^{\frac{n}{p}+s}}, \quad \text{a.e. } (x, y) \in \mathbb{R}^{2n}.$$

Observe that the $L^p(\mathbb{R}^{2n})$ -norm of $S(f)$ coincides with $|f|_{W^{s,p}(G)}$, compare to definition (2.5). The key step is to show that $S(M_G f)(x, y)$ is pointwise almost everywhere dominated by

$$C(n, p, s) \sum_{i,j,k,l \in \{0,1\}} (M_{ij}(M_{kl}(Sf))(x, y) + M_{ij}(M_{kl}(Sf))(y, x)),$$

where each M_{ij} and M_{kl} is either $F \mapsto |F|$ or a V -directional maximal operator in \mathbb{R}^{2n} that is defined in terms of a fixed n -dimensional subspace $V \subset \mathbb{R}^{2n}$, we refer to Definition (2.8). The geometry of the open set G does not have a pivotal role, hence, we are able to prove the pointwise domination without imposing additional restrictions on G . Theorem 1.1 is then a consequence of the fact that the compositions $M_{ij}M_{kl}$ are bounded on $L^p(\mathbb{R}^{2n})$ if $1 < p < \infty$. The described transference of the problem to the $2n$ -dimensional Euclidean space is a typical step when dealing with norm estimates for the spaces $W^{s,p}(G)$, we refer to [4, 6, 21] for other examples. We plan to adapt the transference method to norm estimates on intrinsically defined Triebel–Lizorkin and Besov function spaces on open sets, [20].

As an application of our main result, Theorem 1.1, we study fractional Hardy inequalities. Let us recall that an open set $\emptyset \neq G \subsetneq \mathbb{R}^n$ admits an (s, p) -Hardy inequality, for $0 < s < 1$ and $1 < p < \infty$, if there exists a constant $C > 0$ such that inequality

$$\int_G \frac{|f(x)|^p}{\text{dist}(x, \partial G)^{sp}} dx \leq C \int_G \int_G \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dy dx \quad (1.2)$$

holds for all functions $f \in C_c(G)$. These inequalities have attracted some interest recently, we refer to [2, 3, 4, 6, 7, 8] and the references therein.

In Theorem 4.3 we answer a question from [2], i.e., we characterize those bounded open sets which admit an (s, p) -Hardy inequality. The characterization is given in terms of a localized Maz'ya-type testing condition, where a lower bound $\ell(Q)^{n-sp} \lesssim \text{cap}_{s,p}(Q, G)$ for the fractional (s, p) -capacities of all Whitney cubes $Q \in \mathcal{W}(G)$ is required and a quasiadditivity property of the same capacity is assumed with respect to all finite families of Whitney cubes. Aside from inequality (1.1) an important ingredient in the proof of Theorem 4.3 is the estimate

$$\int_{2^{-1}Q} f dx \leq C \inf_Q M_G f, \quad (1.3)$$

which holds for a constant $C > 0$ that is independent of both $Q \in \mathcal{W}(G)$ and $f \in C_c(G)$. Inequality (1.3) allows us to circumvent the (apparently unknown) weak Harnack inequalities for the minimizers that are associated with the (s, p) -capacities. The weak Harnack based approach is taken up in [14]; therein the counterpart of Theorem 4.3 is obtained in case of the classical Hardy inequality, i.e., for the gradient instead of the fractional Sobolev seminorm.

The structure of this paper is as follows. In Section 2 we present the notation and recall various maximal operators. The proof of Theorem 1.1 is taken up in Section 3. Finally, in Section 4, we give an application of our main result by characterizing fractional (s, p) -Hardy inequalities on bounded open sets.

2. NOTATION AND PRELIMINARIES

Notation. The open ball centered at $x \in \mathbb{R}^n$ and with radius $r > 0$ is written as $B(x, r)$. The Euclidean distance from $x \in \mathbb{R}^n$ to a set E in \mathbb{R}^n is written as $\text{dist}(x, E)$. The Euclidean diameter of E is $\text{diam}(E)$. The Lebesgue n -measure of a measurable set E is denoted by $|E|$. The characteristic function of a set E is written as χ_E . We write $f \in C_c(G)$ if $f : G \rightarrow \mathbb{R}$ is a continuous function with compact support in an open set G . We let $C(\star, \dots, \star)$ denote a positive constant which depends on the quantities appearing in the parentheses only.

For an open set $\emptyset \neq G \subsetneq \mathbb{R}^n$ in \mathbb{R}^n , we let $\mathcal{W}(G)$ be its Whitney decomposition. For the properties of Whitney cubes we refer to [19, VI.1]. In particular, we need the inequalities

$$\text{diam}(Q) \leq \text{dist}(Q, \partial G) \leq 4\text{diam}(Q), \quad Q \in \mathcal{W}(G). \quad (2.4)$$

The center of a cube $Q \in \mathcal{W}(G)$ is written as x_Q and $\ell(Q)$ is its side length. By tQ , $t > 0$, we mean a cube whose sides are parallel to those of Q and that is centered at x_Q and whose side length is $t\ell(Q)$.

Let G be an open set in \mathbb{R}^n . Let $1 < p < \infty$ and $0 < s < 1$ be given. We write

$$|f|_{W^{s,p}(G)} = \left(\int_G \int_G \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dy dx \right)^{1/p} \quad (2.5)$$

for measurable functions f on G that are finite almost everywhere. By $W^{s,p}(G)$ we mean the fractional Sobolev space of functions f in $L^p(G)$ with

$$\|f\|_{W^{s,p}(G)} = \|f\|_{L^p(G)} + |f|_{W^{s,p}(G)} < \infty.$$

Maximal operators. Let $\emptyset \neq G \subsetneq \mathbb{R}^n$ be an open set. The local Hardy–Littlewood maximal function of $f \in L^1_{\text{loc}}(G)$ is defined as follows. For every $x \in G$, we write

$$M_G f(x) = \sup_r \int_{B(x,r)} |f(y)| dy, \quad (2.6)$$

where the supremum ranges over $0 < r < \text{dist}(x, \partial G)$. For notational convenience, we write

$$\int_{B(x,0)} |f(y)| dy = |f(x)| \quad (2.7)$$

whenever $x \in G$ is a Lebesgue point of $|f|$. It is clear that, at the Lebesgue points of $|f|$, the supremum in (2.6) can equivalently be taken over $0 \leq r \leq \text{dist}(x, \partial G)$.

The following lemma is from [2, Lemma 2.3].

Lemma 2.1. *Let $\emptyset \neq G \subsetneq \mathbb{R}^n$ be an open set and $f \in C_c(G)$. Then $M_G f$ is continuous on G .*

Let us fix $i, j \in \{0, 1\}$ and $1 < p < \infty$. For a function $F \in L^p(\mathbb{R}^{2n})$ we write

$$M_{ij}(F)(x, y) = \sup_{r>0} \int_{B(0,r)} |F(x + iz, y + jz)| dz \quad (2.8)$$

for almost every $(x, y) \in \mathbb{R}^{2n}$. Observe that $M_{00}(F) = |F|$. By applying Fubini's theorem in suitable coordinates and boundedness of the centred Hardy–Littlewood maximal operator in $L^p(\mathbb{R}^n)$ we find that $M_{ij} = F \mapsto M_{ij}(F)$ is a bounded operator on $L^p(\mathbb{R}^{2n})$; let us remark that the measurability of $M_{ij}(F)$ for a given $F \in L^p(\mathbb{R}^{2n})$ can be checked by first noting that the supremum in (2.8) can be restricted to the rational numbers $r > 0$ and then adapting the proof of [18, Theorem 8.14] with each r separately.

3. THE PROOF OF THEOREM 1.1

Within this section we prove our main result, namely Theorem 1.1 that is stated in the Introduction. Let us first recall a convenient notation. Namely, for $f \in L^p(G)$ we write

$$S(f)(x, y) = S_{G,n,s,p}(f)(x, y) = \frac{\chi_G(x)\chi_G(y)|f(x) - f(y)|}{|x - y|^{\frac{n}{p}+s}}$$

for almost every $(x, y) \in \mathbb{R}^{2n}$. The main tool for proving Theorem 1.1 is a pointwise inequality, stated in Proposition 3.1, which might be of independent interest.

Proposition 3.1. *Let $\emptyset \neq G \subsetneq \mathbb{R}^n$ be an open set, $0 < s < 1$ and $1 < p < \infty$. Then there exists a constant $C = C(n, p, s) > 0$ such that, for almost every $(x, y) \in \mathbb{R}^{2n}$, inequality*

$$S(M_G f)(x, y) \leq C \sum_{i,j,k,l \in \{0,1\}} (M_{ij}(M_{kl}(Sf))(x, y) + M_{ij}(M_{kl}(Sf))(y, x)) \quad (3.9)$$

holds whenever $f \in L^p(G)$ and $Sf \in L^p(\mathbb{R}^{2n})$.

By postponing the proof of Proposition 3.1 for a while, we can prove Theorem 1.1.

Proof of Theorem 1.1. Fix $f \in L^p(G)$. Without loss of generality, we may assume that the right hand side of inequality (1.1) is finite. Hence $Sf \in L^p(\mathbb{R}^{2n})$ and inequality (1.1) is a consequence of Proposition 3.1 and the boundedness of maximal operators M_{ij} on $L^p(\mathbb{R}^{2n})$. \square

We proceed to the postponed proof that is motivated by that of [16, Theorem 3.2].

Proof of Proposition 3.1. By replacing the function f with $|f|$ we may assume that $f \geq 0$. Since $f \in L^p(G)$ and, hence, $M_G f \in L^p(G)$ we may restrict ourselves to points $(x, y) \in G \times G$ for which both x and y are Lebesgue points of f and both $M_G f(x)$ and $M_G f(y)$ are finite. Moreover, by symmetry, we may further assume that $M_G f(x) > M_G f(y)$. These reductions allow us to find $0 \leq r(x) \leq \text{dist}(x, \partial G)$ and $0 \leq r(y) \leq \text{dist}(y, \partial G)$ such that the estimate

$$\begin{aligned} S(M_G f)(x, y) &= \frac{|M_G f(x) - M_G f(y)|}{|x - y|^{\frac{n}{p}+s}} \\ &= \frac{|\int_{B(x, r(x))} f - \int_{B(y, r(y))} f|}{|x - y|^{\frac{n}{p}+s}} \leq \frac{|\int_{B(x, r(x))} f - \int_{B(y, r_2)} f|}{|x - y|^{\frac{n}{p}+s}} \end{aligned}$$

is valid for any given number

$$0 \leq r_2 \leq \text{dist}(y, \partial G);$$

this number will be chosen in a convenient manner in the two case studies below.

Case $r(x) \leq |x - y|$. Let us denote $r_1 = r(x)$ and choose

$$r_2 = 0. \quad (3.10)$$

If $r_1 = 0$, then we get from (3.10) and our notational convention (2.7) that

$$S(M_G f)(x, y) \leq S(f)(x, y).$$

Suppose then that $r_1 > 0$. Now

$$\begin{aligned} S(M_G f)(x, y) &\leq \frac{1}{|x - y|^{\frac{n}{p} + s}} \left| \int_{B(x, r_1)} f(z) dz - \int_{B(y, r_2)} f(z) dz \right| \\ &= \frac{1}{|x - y|^{\frac{n}{p} + s}} \left| \int_{B(x, r_1)} f(z) - f(y) dz \right| \\ &\lesssim \int_{B(0, r_1)} \frac{\chi_G(x + z) \chi_G(y) |f(x + z) - f(y)|}{|x + z - y|^{\frac{n}{p} + s}} dz \leq M_{10}(Sf)(x, y). \end{aligned}$$

We have shown that

$$S(M_G f)(x, y) \lesssim S(f)(x, y) + M_{10}(Sf)(x, y)$$

and it is clear that inequality (3.9) follows (recall that M_{00} is the identity operator when restricted to non-negative functions).

Case $r(x) > |x - y|$. Let us denote $r_1 = r(x) > 0$ and choose

$$r_2 = r(x) - |x - y| > 0.$$

We then have

$$\begin{aligned} &\left| \int_{B(x, r_1)} f(z) dz - \int_{B(y, r_2)} f(z) dz \right| = \left| \int_{B(0, r_1)} f(x + z) - f(y + \frac{r_2}{r_1} z) dz \right| \\ &= \left| \int_{B(0, r_1)} \left(f(x + z) - \int_{B(y + \frac{r_2}{r_1} z, 2|x - y|) \cap G} f(a) da \right) \right. \\ &\quad \left. + \left(\int_{B(y + \frac{r_2}{r_1} z, 2|x - y|) \cap G} f(a) da - f(y + \frac{r_2}{r_1} z) \right) dz \right| \\ &\leq A_1 + A_2, \end{aligned}$$

where we have written

$$\begin{aligned} A_1 &= \int_{B(0, r_1)} \left(\int_{B(y + \frac{r_2}{r_1} z, 2|x - y|) \cap G} |f(x + z) - f(a)| da \right) dz, \\ A_2 &= \int_{B(0, r_1)} \left(\int_{B(y + \frac{r_2}{r_1} z, 2|x - y|) \cap G} |f(y + \frac{r_2}{r_1} z) - f(a)| da \right) dz. \end{aligned}$$

We estimate both of these terms separately, but first we need certain auxiliary estimates.

Recall that $r_2 = r_1 - |x - y|$. Hence, for every $z \in B(0, r_1)$,

$$\begin{aligned} |y + \frac{r_2}{r_1} z - (x + z)| &= |y - x + \frac{(r_2 - r_1)}{r_1} z| \\ &\leq |y - x| + \frac{|x - y|}{r_1} |z| \leq 2|y - x|. \end{aligned}$$

This, in turn, implies that

$$B(y + \frac{r_2}{r_1} z, 2|x - y|) \subset B(x + z, 4|x - y|) \quad (3.11)$$

whenever $z \in B(0, r_1)$. Moreover, since $r_1 > |x - y|$ and $\{y + \frac{r_2}{r_1}z, x + z\} \subset B(x, r_1) \subset G$ if $|z| < r_1$, we obtain the two equivalences

$$|B(y + \frac{r_2}{r_1}z, 2|x - y|) \cap G| \simeq |x - y|^n \simeq |B(x + z, 4|x - y|) \cap G| \quad (3.12)$$

for every $z \in B(0, r_1)$. Here the implied constants depend only on n .

An estimate for A_1 . The inclusion (3.11) and inequalities (3.12) show that, in the definition of A_1 , we can replace the domain of integration in the inner integral by $B(x + z, 4|x - y|) \cap G$ and, at the same time, control the error term while integrating on average. That is to say,

$$A_1 \lesssim \int_{B(0, r_1)} \left(\int_{B(x+z, 4|x-y|) \cap G} |f(x+z) - f(a)| da \right) dz.$$

By observing that both $x + z$ and a in the last double integral belong to G and using (3.12) again, we can continue as follows:

$$\begin{aligned} \frac{A_1}{|x - y|^{\frac{n}{p}+s}} &\lesssim \int_{B(0, r_1)} \left(\int_{B(x+z, 4|x-y|)} \frac{\chi_G(x+z)\chi_G(a)|f(x+z) - f(a)|}{|x+z-a|^{\frac{n}{p}+s}} da \right) dz \\ &\lesssim \int_{B(0, r_1)} \left(\int_{B(y+z, 5|x-y|)} S(f)(x+z, a) da \right) dz. \end{aligned}$$

Applying the maximal operators defined in Section 2 we find that

$$\frac{A_1}{|x - y|^{\frac{n}{p}+s}} \lesssim \int_{B(0, r_1)} M_{01}(Sf)(x+z, y+z) dz \leq M_{11}(M_{01}(Sf))(x, y).$$

An estimate for A_2 . We use the inclusion $y + \frac{r_2}{r_1}z \in G$ for all $z \in B(0, r_1)$ and then apply the first equivalence in (3.12) to obtain

$$\begin{aligned} A_2 &= \int_{B(0, r_1)} \left(\int_{B(y+\frac{r_2}{r_1}z, 2|x-y|) \cap G} \chi_G(y+\frac{r_2}{r_1}z)\chi_G(a)|f(y+\frac{r_2}{r_1}z) - f(a)| da \right) dz \\ &\lesssim \int_{B(0, r_1)} \left(\int_{B(y+\frac{r_2}{r_1}z, 2|x-y|)} \chi_G(y+\frac{r_2}{r_1}z)\chi_G(a)|f(y+\frac{r_2}{r_1}z) - f(a)| da \right) dz. \end{aligned}$$

Hence, a change of variables yields

$$\begin{aligned} \frac{A_2}{|x - y|^{\frac{n}{p}+s}} &\lesssim \int_{B(0, r_2)} \left(\int_{B(y+z, 2|x-y|)} \frac{\chi_G(y+z)\chi_G(a)|f(y+z) - f(a)|}{|y+z-a|^{\frac{n}{p}+s}} da \right) dz \\ &\lesssim \int_{B(0, r_2)} \left(\int_{B(x+z, 3|x-y|)} S(f)(y+z, a) da \right) dz. \end{aligned}$$

Applying operators M_{01} and M_{11} from Section 2, we can proceed as follows

$$\frac{A_2}{|x - y|^{\frac{n}{p}+s}} \lesssim \int_{B(0, r_2)} M_{01}(Sf)(y+z, x+z) dz \leq M_{11}(M_{01}(Sf))(y, x).$$

Combining the above estimates for A_1 and A_2 we end up with

$$S(M_G f)(x, y) \leq \frac{A_1 + A_2}{|x - y|^{\frac{n}{p}+s}} \lesssim M_{11}(M_{01}(Sf))(x, y) + M_{11}(M_{01}(Sf))(y, x)$$

and inequality (3.9) follows. \square

4. APPLICATION TO FRACTIONAL HARDY INEQUALITIES

We apply Theorem 1.1 by solving a certain localisation problem for (s, p) -Hardy inequalities and our result is formulated in Theorem 4.3 below. Recall that an open set $\emptyset \neq G \subsetneq \mathbb{R}^n$ admits an (s, p) -Hardy inequality, for $0 < s < 1$ and $1 < p < \infty$, if there is a constant $C > 0$ such that inequality

$$\int_G \frac{|f(x)|^p}{\text{dist}(x, \partial G)^{sp}} dx \leq C \int_G \int_G \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dy dx \quad (4.13)$$

holds for all functions $f \in C_c(G)$. We need a characterization of (s, p) -Hardy inequality in terms of the following (s, p) -capacities of compact sets $K \subset G$; we write

$$\text{cap}_{s,p}(K, G) = \inf_u |u|_{W^{s,p}(G)}^p,$$

where the infimum is taken over all real-valued functions $u \in C_c(G)$ such that $u(x) \geq 1$ for every $x \in K$. The ‘Maz’ya-type characterization’ stated in Theorem 4.1 is [2, Theorem 1.1] and it extends to the case $0 < p < \infty$. For information on characterizations of this type, we refer to [17, Section 2] and [10].

Theorem 4.1. *Let $0 < s < 1$ and $1 < p < \infty$. Then an open set $\emptyset \neq G \subsetneq \mathbb{R}^n$ admits an (s, p) -Hardy inequality if and only if there is a constant $C > 0$ such that*

$$\int_K \text{dist}(x, \partial G)^{-sp} dx \leq C \text{cap}_{s,p}(K, G) \quad (4.14)$$

for every compact set $K \subset G$.

We solve a ‘localisation problem of the testing condition (4.14)’, which is stated as a question in [2, p. 2]. Roughly speaking, we prove that if $\text{cap}_{s,p}(\cdot, G)$ satisfies a quasiadditivity property, see Definition 4.2, then G admits an (s, p) -Hardy inequality if and only if inequality (4.14) holds for all Whitney cubes $K = Q \in \mathcal{W}(G)$.

Definition 4.2. *The (s, p) -capacity $\text{cap}_{s,p}(\cdot, G)$ is weakly $\mathcal{W}(G)$ -quasiadditive, if there exists a constant $N > 0$ such that*

$$\sum_{Q \in \mathcal{W}(G)} \text{cap}_{s,p}(K \cap Q, G) \leq N \text{cap}_{s,p}(K, G) \quad (4.15)$$

whenever $K = \bigcup_{Q \in \mathcal{E}} Q$ and $\mathcal{E} \subset \mathcal{W}(G)$ is a finite family of Whitney cubes.

More precisely, we prove the following characterization.

Theorem 4.3. *Let $0 < s < 1$ and $1 < p < \infty$ be such that $sp < n$. Suppose that $G \neq \emptyset$ is a bounded open set in \mathbb{R}^n . Then the following conditions (A) and (B) are equivalent.*

- (A) G admits an (s, p) -Hardy inequality;
- (B) $\text{cap}_{s,p}(\cdot, G)$ is weakly $\mathcal{W}(G)$ -quasiadditive and there exists a constant $c > 0$ such that

$$\ell(Q)^{n-sp} \leq c \text{cap}_{s,p}(Q, G) \quad (4.16)$$

for every $Q \in \mathcal{W}(G)$.

Before the proof of Theorem 4.3, let us make a remark concerning condition (B).

Remark 4.4. *The counterexamples in [2, Section 6] show that neither one of the two conditions (i.e., weak $\mathcal{W}(G)$ -quasiadditivity of the capacity and the lower bound (4.16) for the capacities of Whitney cubes) appearing in Theorem 4.3(B) is implied by the other one. Accordingly, both of these conditions are needed for the characterization.*

Proof of Theorem 4.3. The implication from (A) to (B) follows from [2, Proposition 4.1] in combination with [2, Lemma 2.1]. In the following proof of the implication from (B) to (A) we adapt the argument given in [2, Proposition 5.1].

By Theorem 4.1, it suffices to show that

$$\int_K \text{dist}(x, \partial G)^{-sp} dx \lesssim \text{cap}_{s,p}(K, G), \quad (4.17)$$

whenever $K \subset G$ is compact. Let us fix a compact set $K \subset G$ and an admissible test function u for $\text{cap}_{s,p}(K, G)$. We partition $\mathcal{W}(G)$ as $\mathcal{W}_1 \cup \mathcal{W}_2$, where

$$\mathcal{W}_1 = \{Q \in \mathcal{W}(G) : \langle u \rangle_{2^{-1}Q} := \oint_{2^{-1}Q} u < 1/2\}, \quad \mathcal{W}_2 = \mathcal{W}(G) \setminus \mathcal{W}_1.$$

Write the left-hand side of (4.17) as

$$\left\{ \sum_{Q \in \mathcal{W}_1} + \sum_{Q \in \mathcal{W}_2} \right\} \int_{K \cap Q} \text{dist}(x, \partial G)^{-sp} dx. \quad (4.18)$$

To estimate the first series we observe that, for every $Q \in \mathcal{W}_1$ and every $x \in K \cap Q$,

$$\frac{1}{2} = 1 - \frac{1}{2} < u(x) - \langle u \rangle_{2^{-1}Q} = |u(x) - \langle u \rangle_{2^{-1}Q}|.$$

Thus, by Jensen's inequality and (2.4),

$$\begin{aligned} \sum_{Q \in \mathcal{W}_1} \int_{K \cap Q} \text{dist}(x, \partial G)^{-sp} dx &\lesssim \sum_{Q \in \mathcal{W}_1} \ell(Q)^{-sp} \int_Q |u(x) - \langle u \rangle_{2^{-1}Q}|^p dx \\ &\lesssim \sum_{Q \in \mathcal{W}_1} \ell(Q)^{-n-sp} \int_Q \int_Q |u(x) - u(y)|^p dy dx \\ &\lesssim \sum_{Q \in \mathcal{W}_1} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dy dx \\ &\lesssim |u|_{W^{s,p}(G)}^p. \end{aligned}$$

Let us then focus on the remaining series in (4.18). Let us consider $Q \in \mathcal{W}_2$ and $x \in Q$. Observe that $2^{-1}Q \subset B(x, \frac{4}{5}\text{diam}(Q))$. Hence, by inequalities (2.4),

$$M_G u(x) \gtrsim \oint_{2^{-1}Q} u(y) dy \geq \frac{1}{2}. \quad (4.19)$$

The support of $M_G u$ is a compact set in G by the boundedness of G and the fact that $u \in C_c(G)$. By Lemma 2.1, we find that $M_G u$ is continuous. Concluding from these remarks we find that there is $\rho > 0$, depending only on n , such that $\rho M_G u$ is an admissible test function

for $\text{cap}_{s,p}(\cup_{Q \in \mathcal{W}_2} Q, G)$. The family \mathcal{W}_2 is finite, as $u \in C_c(G)$. Hence, by condition (B) and the inequality (4.19),

$$\begin{aligned} \sum_{Q \in \mathcal{W}_2} \int_{K \cap Q} \text{dist}(x, \partial G)^{-sp} dx &\lesssim \sum_{Q \in \mathcal{W}_2} \ell(Q)^{n-sp} \\ &\leq c \sum_{Q \in \mathcal{W}_2} \text{cap}_{s,p}(Q, G) \\ &\leq cN \text{cap}_{s,p}\left(\bigcup_{Q \in \mathcal{W}_2} Q, G\right) \\ &\leq cN \rho^p \int_G \int_G \frac{|M_G u(x) - M_G u(y)|^p}{|x - y|^{n+sp}} dy dx. \end{aligned}$$

By Theorem 1.1, the last term is dominated by

$$C(n, s, p, N, c, \rho) |u|_{W^{s,p}(G)}^p.$$

The desired inequality (4.17) follows from the considerations above. \square

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(H.L.) DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. Box 35, FI-40014 UNIVERSITY OF JYVÄSKYLÄ, FINLAND

E-mail address: hannes.s.luiro@jyu.fi

(A.V.V) DEPARTMENT OF MATHEMATICS AND STATISTICS, GUSTAF HÄLLSTRÖMIN KATU 2B, FI-00014 UNIVERSITY OF HELSINKI, FINLAND

E-mail address: antti.vahakangas@helsinki.fi